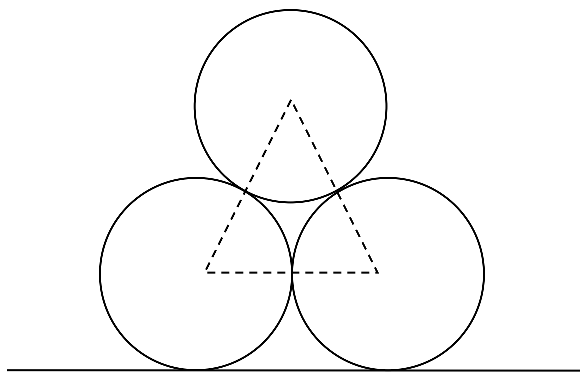
Teacher notes
Topic A

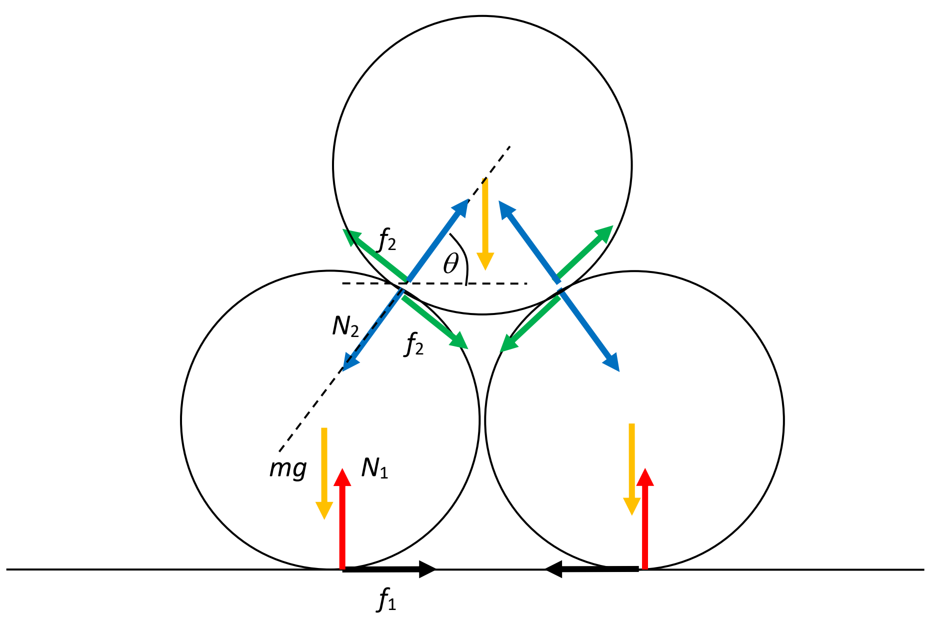
The three-cylinder problem. (A difficult problem for a dark and rainy day.)

Three identical cylinders are stacked as shown. There is friction between the cylinders as well as between the cylinders and the ground.



To keep the problem as simple as possible assume that the two lower cylinders do not touch but are very close to each other.

What is the minimum coefficient of static friction with the ground for equilibrium?



We have the obvious normal forces N_1, N_2 and the weight *mg* of each cylinder. The forces on the right lower cylinder are identical to those on the lower left cylinder by symmetry. The horizontal component of the force N_2 pushes the left cylinder to the left so a frictional force f_1 develops to the right (see comment at the end of this note). This frictional force tends to rotate the lower left cylinder counterclockwise, so a frictional force f_2 develops between this cylinder and the top cylinder. Forces in blue and green are action-reaction pairs. The angle θ is 60°.

Right away we see that we need friction with the ground to have equilibrium. We also need friction between the cylinders otherwise the cylinders would be rotating.

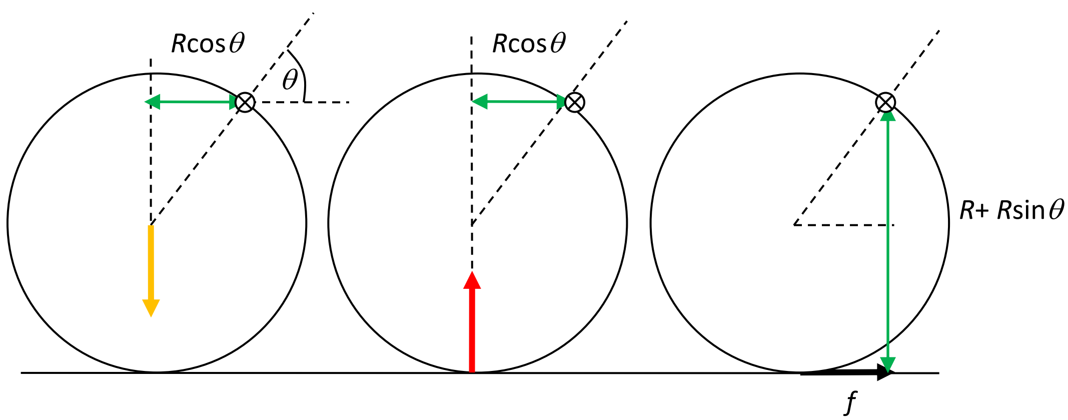
Treating all cylinders as one body we see that the downward force is $3mg$ and the upward is $2N_1$, so equilibrium demands that $2N_1 = 3mg$ and so $N_1 = \frac{3mg}{2}$.

Taking torques about the center of the lower left cylinder gives $f_1 R = f_2 R$, so we learn that $f_1 = f_2 = f$.

Taking torques about the point where the lower left cylinder touches the top cylinder gives:

$$mgR\cos\theta + f(R + R\sin\theta) = N_1 R\cos\theta$$

The distances needed for calculating torques are shown in the figure:



Therefore, using $N_1 = \frac{3mg}{2}$ we find $mg\cos\theta + f(1+\sin\theta) = \frac{3mg}{2}\cos\theta$ i.e.

$$f = \frac{mg}{2}\frac{\cos\theta}{1+\sin\theta} = \frac{mg}{2}\frac{\cos 60^\circ}{1+\sin 60^\circ} = \frac{mg}{2}\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}} = \frac{mg}{4}\frac{2}{2+\sqrt{3}} = \frac{mg}{2}\frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} = \frac{mg}{2}(2-\sqrt{3}).$$

The minimum coefficient of friction will require the maximum frictional force and so $f = \mu N_1 = \mu \dfrac{3mg}{2}$.

Hence, $\dfrac{mg}{2}(2-\sqrt{3}) = \mu \dfrac{3mg}{2}$ and so

$$\mu = \frac{2-\sqrt{3}}{3} \approx 0.0893 \text{ .}$$

(Taking horizontal force components on the lower left cylinder gives:

$$N_2 \cos\theta = f + f\sin\theta \Rightarrow N_2 = f\frac{1+\sin\theta}{\cos\theta} = \frac{mg}{2}\frac{\cos\theta}{1+\sin\theta} \times \frac{1+\sin\theta}{\cos\theta} = \frac{mg}{2} \text{ .}$$

This means that the net horizontal force component on the lower left cylinder at the point where it touches the top cylinder is to the left, justifying the original claim that the frictional force with the ground is pointing to the right.)